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Application of Rigid Finite Element Method to Dynamic Analysis of Spatial Systems

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This paper presents an application of the rigid finite element method to modeling of flexible links of spatial systems. It is assumed that the movement of the base, with which the flexible system is connected, is known. The model presented takes into account large deflections of the link and an influence of centrifugal forces on deflections and deformations. Methods applied in dynamic analysis of manipulators with rigid links are used to derive the equations of motion. The results of numerical calculations are compared with those obtained by other authors who used the finite element approach.

Nomenclature

a_j	= length of link
$c\alpha$, $s\alpha$	= $\cos\alpha$, $\sin\alpha$
c_j^e, k_j^e	= coefficients of translation and rotational stiffness of SDE
D	= function of energy dissipation
d_i	= displacement of joint
$\boldsymbol{F}_i$, $\boldsymbol{F}$	= force vectors of frame $\{i\}$ and of system
$\boldsymbol{I}$	= identity matrix
$\boldsymbol{K}_i$, $\boldsymbol{K}$	= stiffness matrix of frame $\{i\}$ and of system
$\boldsymbol{L}_i$, $\boldsymbol{L}$	= dumping matrix of frame $\{i\}$ and of system
$\boldsymbol{M}_i$, $\boldsymbol{M}$	= mass matrix of frame $\{i\}$ and of system
m_i	= mass of RFE
$\boldsymbol{Q}_i$, $\boldsymbol{Q}$	= components and vector of generalized forces resulting from external loads
$\boldsymbol{q}_i$, $\boldsymbol{q}$	= vector of generalized displacement of frame $\{i\}$ and of system
${}^j\boldsymbol{r}_i$	= coordinate vector of point belonging to ith link with respect to $\{j\}$ frame
T	= kinetic energy of the system
${}^j_i\boldsymbol{T}$, $\boldsymbol{A}_0$, $\boldsymbol{B}_i$	= transformation matrices [e.g., from $\{i\}$ to $\{j\}$ frame]
$\mathrm{tr}(\boldsymbol{A})$	= trace of $\boldsymbol{A}$ matrix
V	= potential energy of system
x_{ij}, φ_{ij}	= generalized coordinates of ith RFE, $j = 1, 2, 3$
α_i	= twist of link
θ_i	= angle of joint
$\Omega(\boldsymbol{q})$	= vector with elements that are quadratic forms of q

Superscript

*	= transposition of vector or matrix

Introduction

ONE of the important tasks of present-day mechanics is elaboration of methods for computer simulation of the dynamics of machines and mechanisms with flexible links. An increase of machine working speeds together with the simultaneous desire or light constructions that are economical in energy terms means that methods considering flexibility of links are sought more and more often.

An essential feature of models created for the purpose of machine design and construction is the possibility of changing the configuration caused by the base motion. The changing configuration causes additional inertial forces and vibrations of flexible links. There is also an opposite connection. Vibrations of flexible links can influence disturbances of the base motion. Thus, changing the configuration presents a major difficulty in modeling of machines and requires more complicated models than those used for classical vibration problems, met, for example, in civil engineering.

This paper has been inspired by papers of Kane et al.[1] and Du et al.[2] that present linear and nonlinear models for vibration analysis of beams attached to a moving base undergoing a known motion. In the above papers the finite element method (FEM) is used in order to discretize the flexible beam. The results of calculations are presented and conclusions on the range of the use of both linear and nonlinear models are formulated.

In this paper, the rigid finite element method (RFEM) by Kruszewski et al.[3] and Gawronski et al.[4] is applied to discretization of the flexible beam. As in the above papers, it is assumed that the motion of the base is known. The equations of motion are developed based on methods used in dynamic analysis of manipulators with rigid links by Craig[5] and Jurewicz.[6] Since the RFEM is known mainly in Poland, a brief description is given below.

Rigid Finite Element Method

This method for analysis of vibrations about the static equilibrium of plane and spatial systems has been developed and applied in the Department of Mechanics and Strength of Materials of the Technical University of Gdansk for the last 20 years and is reflected in monographs.[3,4]

In this method any link (continuum body) can be divided into rigid finite elements (RFEs) characterized by their masses and inertial moments that are connected by nondimensional and massless spring-damping elements (SDEs) (Fig. 1). The work[3] gives more detailed information about dividing beam and plate elements into RFEs and SDEs. The method has successfully been used for modeling of

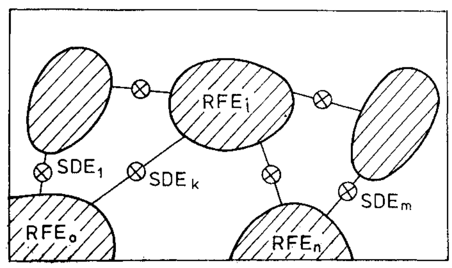

Fig. 1a System of RFEs connected with SDEs.

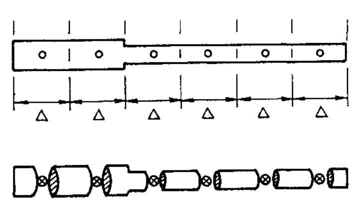

Fig. 1b Example of division of beam into RFEs and SDEs.

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vibrations of such complex systems as ship vessels, ship propulsion systems, drive systems, and cranes. However, consideration in these works has been limited to analysis of vibrations about the static equilibrium.

The formulation of the method for dynamic analysis of planar systems with changing configuration is presented in Ref. 7. In this case, a small motion (relative motion) that results from deformations of subsystems is imposed on the large motion of subsystems (base motion). Some modification of the RFEM for modeling of flexible links of planar linkage mechanisms is presented in Ref. 8. The method proposed enables consideration of large deflections of flexible links.

Generalization of this modification for spatial systems in an application to robot manipulators is described in Ref. 9. Reference 10 describes its further development and presents, inter alia, a comparison of calculation and experimental results of spatial vibrations with large amplitudes of a cantilever beam. Results of research on the application of the modification to analysis of viscoelastic beam vibrations with large amplitudes are also presented in Refs. 11 and 12.

In this context it is useful to refer to Wittenburg.[13] In this paper the formulation of the equations of motion for spatial systems with changing configuration is presented. The author has also allowed for the possibility of connecting rigid bodies with SDEs. Despite external similarity of methods presented in Refs. 3 and 13, they differ radically. Spring-damping elements introduced by Wittenburg serve mainly to model flexible joints, whereas the RFEM is used for modeling of flexible links. Mathematical formalism used for deriving the equations of motion is also completely different. In Ref. 3 the Newton–Euler equations are applied, whereas in works employing the RFEM the equations of motion are usually derived from the Lagrange equations. There is also a similar discrete model used by Banerjee.[14] In this case the discrete model contains also rigid bodies connected by SDEs, but the discretization of continuum and description of the model are different.

Below the equations of motion of a flexible spatial system with a known base motion are derived.

Equations of Motion of Flexible System

As mentioned above, to derive the equations of motion, the methods for kinematic and dynamic analysis of rigid manipulators are used.[5] A relative position of rigid bodies is described by the Denavit–Hartenberg parameters. In the case of Fig. 2, when two rigid bodies are connected so as to form a kinematic pair of the fifth class, their relative position is described by the following parameters: a_{i-1}, α_{i-1}, d_i, and Θ_i.

In the case of a rotary kinematic pair the joint variable is an angle Θ_i whereas a_{i-1}, α_{i-1}, and d_i are constant. However, for a sliding kinematic pair d_i is a joint variable and a_{i-1}, α_{i-1}, and Θ_i are constant.

A frame $\{i\}$ is fixed to each link. Transformation matrices from the $\{i\}$ to the $\{i-1\}$ frame then have the form[5]

$$ {}^{i-1}_{\;\;i}\boldsymbol{T} = \begin{bmatrix} c\Theta_i & -s\Theta_i & 0 & a_{i-1} \\ s\Theta_i c\alpha_{i-1} & c\Theta_i c\alpha_{i-1} & -s\alpha_{i-1} & s\alpha_{i-1} d_i \\ s\Theta_i s\alpha_{i-1} & c\Theta_i c\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1} $$

Transformation of coordinates from the frame $\{i\}$ to the base frame is performed according to the formula (where $\boldsymbol{B}_i = {}^0_1\boldsymbol{T}\cdot{}^1_2\boldsymbol{T}\cdots{}^{i-1}_{\;\;i}\boldsymbol{T}$)

$$ {}^0_i\boldsymbol{r} = \boldsymbol{r}_i = {}^0_1\boldsymbol{T}\cdot{}^1_2\boldsymbol{T}\cdots{}^{i-1}_{\;\;i}\boldsymbol{T}\cdot{}^i\boldsymbol{r}_i = \boldsymbol{B}_i\cdot{}^i\boldsymbol{r}_i \tag{2} $$

Assuming the system consists of $n+1$ RFEs and the RFE_0 (Fig. 1) is a base of the element system considered, coordinate systems can be introduced as in Fig. 3. Further consideration is given to vibrations of the body system, with which the RFE system considered is connected, caused by the transportation motion of the frame $\{r\}$. Knowledge of the transportation motion means that there are known coordinates of the origin of the $\{r\}$ frame as well as angles between

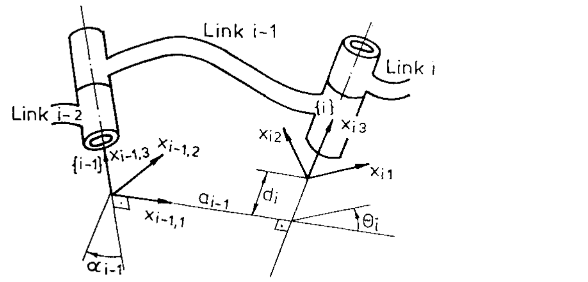

Fig. 2 Parameters describing mutual position of links.

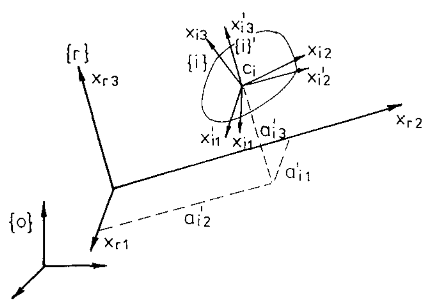

Fig. 3 Coordinate systems: $\{0\}$ is inertial frame, $\{r\}$ is base frame, and $\{i\}$, $\{i'\}$ are local frames fixed to ith RFE (before and after deformation of system, respectively).

the axes of the frame $\{r\}$ fixed to RFE_0 and axes of the inertial frame $\{0\}$.

Thus, the transformation matrix $\boldsymbol{A}_0$ is given as

$$ \boldsymbol{A}_0(t) = {}^0_r\boldsymbol{T}(t) \tag{3} $$

as a function of time independent of generalized coordinates defined below. There are two local coordinate systems fixed to each RFE ($i = 1, \ldots, n$): the frame $\{i\}$ with the origin placed in the mass center and axes covering the principal inertial axes and the frame $\{i'\}$, which covers the position of the frame $\{i\}$ before link deformation.

The position of the RFEs in the undeformed state can be unequivocally defined with reference to the frame $\{r\}$ if the following transformation matrices are known: ${}^r_{i'}\boldsymbol{T} = \mathrm{const}$, $i = 1, 2, \ldots, n$.

Let us assume that on account of base motion or external forces, each RFE experiences small relative deflections in the frame $\{r\}$ that are components of the vector:

$$ \boldsymbol{q}_i = [x_{i1} \quad x_{i2} \quad x_{i3} \quad \varphi_{i1} \quad \varphi_{i2} \quad \varphi_{i3}]^* \tag{4} $$

With this assumption the transformation matrix from the $\{i\}$ to the $\{i'\}$ frame can be written as[15]

$$ {}^{i'}_{i}\boldsymbol{T} = \begin{bmatrix} 1 & -\varphi_{i3} & \varphi_{i2} & x_{i1} \\ \varphi_{i3} & 1 & -\varphi_{i1} & x_{i2} \\ -\varphi_{i2} & \varphi_{i1} & 1 & x_{i3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5} $$

Finally after superposition of transformations, their matrices from the frame $\{i\}$ to the inertial frame take the form

$$ \boldsymbol{B}_i = \boldsymbol{B}_i(t, \boldsymbol{q}_i) = \boldsymbol{A}_0(t)\cdot{}^r_{i'}\boldsymbol{T}\cdot{}^{i'}_{i}\boldsymbol{T}(\boldsymbol{q}_i) \tag{6} $$

Forms of matrices $\boldsymbol{B}_i$ and ${}^{i'}_{i}\boldsymbol{T}$ resulting from the assumption that the angles φ_{i1}, φ_{i2}, and φ_{i3} are small enable the transformation matrices to be written in the following form, which is more convenient for further consideration:

$$ \boldsymbol{B}_i = \boldsymbol{A}_{i0}\cdot\left[\boldsymbol{I} + \sum_{j=1}^{6}\boldsymbol{D}_{ij}\cdot q_{ij}\right] = \boldsymbol{A}_{i0}\cdot\boldsymbol{A}_i \tag{7} $$

where

$$ \boldsymbol{A}_{i0} = \boldsymbol{A}_0\cdot{}^r_{i'}\boldsymbol{T}, \qquad \boldsymbol{A}_i = {}^{i'}_{i}\boldsymbol{T} = \boldsymbol{I} + \sum_{j=1}^{6}\boldsymbol{D}_{ij}\cdot q_{ij} $$

$$D_{i1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_{i2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_{i3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_{i4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_{i5} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_{i6} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$q_{i1} = x_{i1}, \qquad q_{i2} = x_{i2}, \qquad q_{i3} = x_{i3}$$

$$q_{i4} = \varphi_{i1}, \qquad q_{i5} = \varphi_{i2}, \qquad q_{i6} = \varphi_{i3}$$

It can be seen that

$$D_{ij} = \frac{\partial A_i}{\partial q_{ij}} \qquad i = 1, 2, \dots, n, \qquad j = 1, \dots, 6 \tag{8}$$

Since the matrices B_i are defined, the coordinates of any point of RFE$_i$ with relation to the inertial system can be defined according to the formula

$$r_i = B_i \cdot {}^i r_i \tag{9}$$

In the case when axes of frame $\{i'\}$ are parallel to the base frame $\{r\}$, relationships describing elements of B_i matrices are considerably simplified. It occurs that (Fig. 3)

$${}^r_{i'}T = \begin{bmatrix} 1 & 0 & 0 & a_{i'1} \\ 0 & 1 & 0 & a_{i'2} \\ 0 & 0 & 1 & a_{i'3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

and relationship (7) can be expressed as (where $D_{i0} = {}^r_{i'}T$)

$$B_i = A_0 \cdot \left(D_{i0} + \sum_{j=1}^{6} D_{ij} \cdot q_{ij} \right) \tag{11}$$

The equations of motion of the system with changing configuration, when vectors of generalized coordinates of RFE are defined according to Eq. (4), are obtained using Lagrange equations of second order. The equations require the definition of the kinetic (T) and potential (V) energies of the system, the function of dissipation (D), and generalized forces resulting from external loads (Q).

A. Kinetic Energy of System

The kinetic energy of the RFE$_i$ can be expressed as[6]

$$T_i = \frac{1}{2} \int_{m_i} \dot{r}_i^* \cdot \dot{r}_i \, \mathrm{d}m_i = \frac{1}{2} \int_{m_i} \mathrm{tr}(\dot{r}_i \cdot \dot{r}_i^*) \, \mathrm{d}m_i \tag{12}$$

Applying relationship (9), the following can be obtained:

$$T_i = \frac{1}{2} \int_{m_i} \mathrm{tr}\left(\dot{B}_i \cdot {}^i r_i \cdot {}^i r_i^* \cdot \dot{B}_i^*\right) \mathrm{d}m_i$$

$$= \frac{1}{2} \mathrm{tr}\left(\dot{B}_i \cdot \int_{m_i} {}^i r_i \cdot {}^i r_i^* \, \mathrm{d}m_i \cdot \dot{B}_i^* \right) = \frac{1}{2} \mathrm{tr}(\dot{B}_i \cdot H_i \cdot \dot{B}_i^*) \tag{13}$$

where

$$H_i = \int_{m_i} \begin{bmatrix} {}^i x_{i1} & {}^i x_{i2} & {}^i x_{i3} & 1 \end{bmatrix}^* \begin{bmatrix} {}^i x_{i1} & {}^i x_{i2} & {}^i x_{i3} & 1 \end{bmatrix} \mathrm{d}m_i$$

which for the case in which the local frame $\{i\}$ is a system of principal central axes takes the form

$$H_i = \mathrm{diag}[h_{ij}] \tag{14}$$

where $h_{ij} = \int_{m_i} {}^i x_{ij}^2 \, \mathrm{d}m_i$, $j = 1, 2, 3$, and $h_{i4} = \int_{m_i} \mathrm{d}m_i = m_i$.

Thus, applying Eqs. (7–13), the following is obtained:

$$T_i = \tfrac{1}{2} \mathrm{tr}[(\dot{A}_{i0} \cdot A_i + A_{i0} \cdot \dot{A}_i) \cdot H_i \cdot (A_i^* \cdot \dot{A}_{i0}^* + \dot{A}_i^* \cdot A_{i0}^*)]$$

$$= \tfrac{1}{2} \mathrm{tr}(\dot{A}_{i0} \cdot A_i \cdot H_i \cdot A_i^* \cdot \dot{A}_{i0}^* + \dot{A}_{i0} \cdot A_i \cdot H_i \cdot \dot{A}_i^* \cdot A_{i0}^*$$

$$+ A_{i0} \cdot \dot{A}_i \cdot H_i \cdot A_i^* \cdot \dot{A}_{i0}^* + A_{i0} \cdot \dot{A}_i \cdot H_i \cdot \dot{A}_i^* \cdot A_{i0}^*) \tag{15}$$

Using the relationships

$$\mathrm{tr}(A \cdot H \cdot B^*) = \mathrm{tr}(H \cdot B^* \cdot A), \qquad \mathrm{tr}\left(\sum_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mathrm{tr}(A_i)$$

where A_i are square matrices, the expression of the kinetic energy of the RFE can be written in the form

$$T_i = \tfrac{1}{2} \mathrm{tr}(\dot{A}_{i0} \cdot A_i \cdot H_i \cdot A_i^* \cdot \dot{A}_{i0}^*) + \mathrm{tr}(A_{i0} \cdot \dot{A}_i \cdot H_i \cdot A_i^* \cdot \dot{A}_{i0}^*)$$

$$+ \tfrac{1}{2} \mathrm{tr}(A_{i0} \cdot \dot{A}_i \cdot H_i \cdot \dot{A}_i^* \cdot A_{i0}^*) \tag{16}$$

The kinetic energy of the system of nth RFEs is defined as

$$T = \sum_{i=1}^{n} T_i \tag{17}$$

where T_i is defined by Eq. (16).

For further consideration it is useful to define the Lagrange equation component, which after some transformation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = M_i \cdot \ddot{q}_i + B_i \cdot \dot{q}_i + C_i \cdot q_i + F_i \tag{18}$$

where q_i is defined in Eq. (4); M_i, B_i, and C_i are 6×6 square matrices with elements

$$M_{i,pq} = \mathrm{tr}(S_{i,pq} \cdot A_{i0}^* \cdot A_{i0}) = \mathrm{tr}(S_{i,pq} \cdot U_i) \tag{19a}$$

$$B_{i,pq} = 2\,\mathrm{tr}(S_{i,pq} \cdot \dot{A}_{i0}^* \cdot A_{i0}) = 2\,\mathrm{tr}(S_{i,pq} \cdot V_i) \tag{19b}$$

$$C_{i,pq} = \mathrm{tr}(S_{i,pq} \cdot \ddot{A}_{i0}^* \cdot A_{i0}) = \mathrm{tr}(S_{i,pq} \cdot W_i) \tag{19c}$$

and F_i is a vector with elements

$$F_{i,p} = \mathrm{tr}(S_{i,p0} \cdot \ddot{A}_{i0}^* \cdot A_{i0}) = \mathrm{tr}(S_{i,p0} \cdot W_i)$$

$$U_i = A_{i0}^* \cdot A_{i0}, \qquad V_i = \dot{A}_{i0}^* \cdot A_{i0}, \qquad W_i = \ddot{A}_{i0} \cdot A_{i0}$$

$$S_{i,pq} = D_{ip} \cdot H_i \cdot D_{iq}^*, \qquad S_{i,p0} = D_{ip} \cdot H_i \cdot I$$

$$p, q = 1, 2, \dots, 6$$

As can be seen from the above, in order to calculate the elements of matrices M_i, B_i, C_i and vectors F_i, the elements of matrices $S_{i,pq} (p = 1, 2, \dots, 6, q = 0, 1, 2, \dots, 6)$, which are matrices with constant coefficients, and traces of products from Eq. (19) have to be calculated first. In order to use the special forms of these matrices, a direct calculation of their elements has been made. The formulas obtained are presented below:

$$M_i = \mathrm{diag}[h_{i4} \quad h_{i4} \quad h_{i4} \quad h_{i2} + h_{i3} \quad h_{i1} + h_{i3} \quad h_{i1} + h_{i2}] \tag{20}$$

$$
\boldsymbol{B}_i = 2\begin{bmatrix} h_{i4}V_{i11} & h_{i4}V_{i21} & h_{i4}V_{i31} & 0 & 0 & 0 \\ h_{i4}V_{i12} & h_{i4}V_{i22} & h_{i4}V_{i32} & 0 & 0 & 0 \\ h_{i4}V_{i13} & h_{i4}V_{i23} & h_{i4}V_{i33} & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{i3}V_{i22}+h_{i2}V_{i33} & -h_{i3}V_{i12} & -h_{i2}V_{i13} \\ 0 & 0 & 0 & -h_{i3}V_{i21} & h_{i3}V_{i11}+h_{i1}V_{i33} & -h_{i1}V_{i23} \\ 0 & 0 & 0 & -h_{i2}V_{i31} & -h_{i1}V_{i32} & h_{i2}V_{i11}+h_{i1}V_{i22} \end{bmatrix} \tag{21a}
$$

$$
\boldsymbol{C}_i = \begin{bmatrix} h_{i4}W_{i11} & h_{i4}W_{i21} & h_{i4}W_{i31} & 0 & 0 & 0 \\ h_{i4}W_{i12} & h_{i4}W_{i22} & h_{i4}W_{i32} & 0 & 0 & 0 \\ h_{i4}W_{i13} & h_{i4}W_{i23} & h_{i4}W_{i33} & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{i3}W_{i22}+h_{i2}W_{i33} & -h_{i3}W_{i12} & -h_{i2}W_{i13} \\ 0 & 0 & 0 & -h_{i3}W_{i21} & h_{i3}W_{i11}+h_{i1}W_{i33} & -h_{i1}W_{i23} \\ 0 & 0 & 0 & -h_{i2}W_{i31} & -h_{i1}W_{i32} & h_{i2}W_{i11}+h_{i1}W_{i22} \end{bmatrix} \tag{21b}
$$

$$
\boldsymbol{F}_i = [h_{i4}W_{i41} \quad h_{i4}W_{i42} \quad h_{i4}W_{i43} \quad -h_{i3}W_{i32}+h_{i2}W_{i23} \quad h_{i3}W_{i31}-h_{i1}W_{i13} \quad -h_{i2}W_{i21}+h_{i1}W_{i12}] \tag{22}
$$

The formulas given allow us to calculate coefficients from Eq. (19) avoiding repeated multiplications of matrices and calculations of their traces.

If the generalized coordinate vector has the form

$$
\boldsymbol{q} = [\boldsymbol{q}_1^* \quad \boldsymbol{q}_2^* \quad \cdots \quad \boldsymbol{q}_n^*]^* \tag{23}
$$

then, having calculated expressions of $\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i$, and $\boldsymbol{F}_i$, the components of the equations of motion connected with the kinetic energy in Eq. (18) can be defined as

$$
\frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \boldsymbol{M}\cdot\ddot{\boldsymbol{q}} + \boldsymbol{B}\cdot\dot{\boldsymbol{q}} + \boldsymbol{C}\cdot\boldsymbol{q} + \boldsymbol{F} \tag{24}
$$

where $\boldsymbol{M} = \text{diag}[\boldsymbol{M}_1 \ \boldsymbol{M}_2 \ \cdots \ \boldsymbol{M}_n]$ is a matrix with constant coefficients, $\boldsymbol{B}(t) = \text{diag}[\boldsymbol{B}_1 \ \boldsymbol{B}_2 \ \cdots \ \boldsymbol{B}_n]$, $\boldsymbol{C}(t) = \text{diag}[\boldsymbol{C}_1 \ \boldsymbol{C}_2 \ \cdots \ \boldsymbol{C}_n]$, $\boldsymbol{F} = [\boldsymbol{F}_1^* \ \boldsymbol{F}_2^* \ \cdots \ \boldsymbol{F}_n^*]^*$, and $\boldsymbol{M}_i, \boldsymbol{B}_i, \boldsymbol{C}_i, \boldsymbol{F}_i$ are defined by Eqs. (20–22).

B. Potential Energy of System

In order to define remaining components of the equations of motion the elastic strain energy and the function of energy dissipation of SDE ($i = 1, \ldots, m$) have to be calculated.

The potential energy of elastic strain of SDEs can be presented in the form

$$
V = \sum_{e=1}^{m} V_e \tag{25}
$$

Figure 4 shows the SDE_e connecting RFE_l and RFE_p. It is obvious that the strain energy of this element will depend only on coordinates $\boldsymbol{q}_l$ and $\boldsymbol{q}_p$. Let us assume that coordinates of the SES_e with respect to local systems fixed to RFEs for the frames $\{l\}$ and $\{p\}$ are defined by the vectors

$$
{}^l\boldsymbol{r}_e = [\eta_{l1} \quad \eta_{l2} \quad \eta_{l3} \quad 1]^*, \qquad {}^p\boldsymbol{r}_e = [\eta_{p1} \quad \eta_{p2} \quad \eta_{p3} \quad 1]^* \tag{26}
$$

In order to define the elastic strain energy of the SDE_e, the coordinates of the point in which SDE is placed as the coordinates of the point belonging to RFE l and p with respect to the same coordinate system have to be defined. The frame $\{r\}$ fixed to the moving base can be chosen as a joint frame of reference. The coordinates of the SDE_e as the point of the RFE l and p can then be described according to the formulas

$$
{}^r\boldsymbol{r}_{el} = {}^r_{l'}\boldsymbol{T}\cdot{}^{l'}_{l}\boldsymbol{T}\cdot{}^l\boldsymbol{r}_e, \qquad {}^r\boldsymbol{r}_{ep} = {}^r_{p'}\boldsymbol{T}\cdot{}^{p'}_{p}\boldsymbol{T}\cdot{}^p\boldsymbol{r}_e \tag{27}
$$

where ${}^r_{l'}\boldsymbol{T}, {}^r_{p'}\boldsymbol{T}$ are matrices with constant coefficients and ${}^{l'}_{l}\boldsymbol{T}, {}^{p'}_{p}\boldsymbol{T}$ are matrices with elements depending on generalized coordinates of RFE l and p.

Assuming, as in the calculation example considered below, that axes of the frame $\{i'\}$ are parallel to axes of the frame $\{r\}$ fixed to the moving base (RFE_0), the following relationships are obtained:

$$
{}^r_{l'}\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{u}_l \\ \boldsymbol{0} & 1 \end{bmatrix}, \qquad {}^r_{p'}\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{u}_p \\ \boldsymbol{0} & 1 \end{bmatrix} \tag{28}
$$

However, matrices ${}^{i'}_{i}\boldsymbol{T}$ for $i \in \{l, p\}$ can be defined by the formula

$$
{}^{i'}_{i}\boldsymbol{T} = \begin{bmatrix} c\varphi_{i2}c\varphi_{i3} & -c\varphi_{i2}s\varphi_{i3} & s\varphi_{i2} & x_{i1} \\ s\varphi_{i1}s\varphi_{i2}c\varphi_{i3}+c\varphi_{i1}s\varphi_{i3} & -s\varphi_{i1}s\varphi_{i2}s\varphi_{i3}+c\varphi_{i1}c\varphi_{i3} & -s\varphi_{i1}c\varphi_{i2} & x_{i2} \\ -c\varphi_{i1}s\varphi_{i2}c\varphi_{i3}+s\varphi_{i1}s\varphi_{i3} & c\varphi_{i1}s\varphi_{i2}s\varphi_{i3}+s\varphi_{i1}c\varphi_{i3} & c\varphi_{i1}c\varphi_{i2} & x_{i3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad i = 1, 2, \ldots, n \tag{29}
$$

From the above the coordinates of the SDE_e as of the point belonging to RFE_i, where $i \in \{l, p\}$, with respect to the frame $\{r\}$ can be calculated as

$$
x_{i1}^e = c\varphi_{i2}c\varphi_{i3}\eta_{i1} - c\varphi_{i2}s\varphi_{i3}\eta_{i2} + s\varphi_{i2}\eta_{i3} + x_{i1} + u_{i1} \tag{30a}
$$

$$
x_{i2}^e = (s\varphi_{i1}s\varphi_{i2}c\varphi_{i3} + c\varphi_{i1}s\varphi_{i3})\eta_{i1} + (-s\varphi_{i1}s\varphi_{i2}s\varphi_{i3} + c\varphi_{i1}c\varphi_{i2})\eta_{i2} - s\varphi_{i1}c\varphi_{i2}\eta_{i3} + x_{i2} + u_{i2} \tag{30b}
$$

$$
x_{i3}^e = (-c\varphi_{i1}s\varphi_{i2}c\varphi_{i3} + s\varphi_{i1}s\varphi_{i3})\eta_{i1} + (c\varphi_{i1}s\varphi_{i2}s\varphi_{i3} + s\varphi_{i1}c\varphi_{i2})\eta_{i2} - s\varphi_{i1}c\varphi_{i2}\eta_{i3} + x_{i3} + u_{i3} \tag{30c}
$$

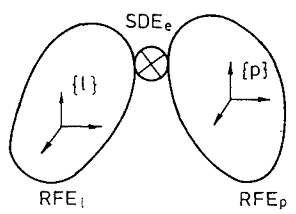

Fig. 4 SDE connecting RFEs.

where ${}^{r}\boldsymbol{r}_{ei} = [x_{i1} \quad x_{i2} \quad x_{i3} \quad 1]^*$ from formula (27) and $\boldsymbol{u}_i = [u_{i1} \quad u_{i2} \quad u_{i3} \quad 1]^*$ from formula (28).

The elastic strain energy of the SDE can be written in the form[3]

$$V_e = \sum_{j=1}^{3} \frac{1}{2} c_j^e \left(x_{pj}^e - x_{lj}^e\right)^2 + \sum_{j=1}^{3} \frac{1}{2} k_j^e \left(\varphi_{pj}^e - \varphi_{lj}^e\right)^2 \tag{31}$$

Thus, the following expressions enter the equations of motion:

$$\frac{\partial V_e}{\partial x_{pj}} = c_j^e \left(x_{pj}^e - x_{lj}^e\right) \tag{32a}$$

$$\frac{\partial V_e}{\partial \varphi_{pj}} = k_j^e \left(\varphi_{pj}^e - \varphi_{lj}^e\right) + \sum_{k=1}^{3} c_k^e \left(x_{pk}^e - x_{lk}^e\right) \frac{\partial x_{pk}^e}{\partial \varphi_{pj}} \tag{32b}$$

$$\frac{\partial V_e}{\partial x_{lj}} = -c_j^e \left(x_{pj}^e - x_{lj}^e\right) \tag{33a}$$

$$\frac{\partial V_e}{\partial \varphi_{lj}} = -k_j^e \left(\varphi_{pj}^e - \varphi_{lj}^e\right) - \sum_{k=1}^{3} c_k^e \left(x_{pk}^e - x_{lk}^e\right) \frac{\partial x_{lk}^e}{\partial \varphi_{lj}} \qquad j = 1, 2, 3 \tag{33b}$$

Now, linearization of the trigonometric functions of the angles φ_{lj} gives

$$x_{i1}^e = \eta_{i1} + u_{i1} + x_{i1} + \eta_{i3}\varphi_{i2} - \eta_{i2}\varphi_{i3} \tag{34a}$$

$$x_{i2}^e = \eta_{i2} + u_{i2} + x_{i2} - \eta_{i3}\varphi_{i1} + \eta_{i1}\varphi_{i3} \tag{34b}$$

$$x_{i3}^e = \eta_{i3} + u_{i3} + x_{i3} + \eta_{i2}\varphi_{i1} - \eta_{i1}\varphi_{i2} \tag{34c}$$

$$\frac{\partial x_{i1}^e}{\partial \varphi_{i1}} = 0, \qquad \frac{\partial x_{i2}^e}{\partial \varphi_{i1}} = -\eta_{i3} - \varphi_{i1}\eta_{i2} + \varphi_{i2}\eta_{i1}, \qquad \frac{\partial x_{i3}^e}{\partial \varphi_{i1}} = \eta_{i2} - \varphi_{i1}\eta_{i3} + \varphi_{i3}\eta_{i1} \tag{35a}$$

$$\frac{\partial x_{i1}^e}{\partial \varphi_{i2}} = \eta_{i3} - \varphi_{i2}\eta_{i1}, \qquad \frac{\partial x_{i2}^e}{\partial \varphi_{i2}} = \varphi_{i1}\eta_{i1}, \qquad \frac{\partial x_{i3}^e}{\partial \varphi_{i2}} = -\eta_{i1} - \varphi_{i2}\eta_{i3} + \varphi_{i3}\eta_{i2} \tag{35b}$$

$$A_0 = \begin{bmatrix} c\psi_1 s\psi_2 & -c\psi_1 c\psi_2 s\psi_3 - s\psi_1 c\psi_3 & c\psi_1 c\psi_2 c\psi_3 - s\psi_1 s\psi_3 & c\psi_1 c\psi_2 L_2 + c\psi_1 L_1 \\ s\psi_1 s\psi_2 & -s\psi_1 c\psi_2 s\psi_3 + c\psi_1 c\psi_3 & s\psi_1 c\psi_2 c\psi_3 + c\psi_1 s\psi_3 & s\psi_1 c\psi_2 L_2 + s\psi_1 L_1 \\ -c\psi_2 & -s\psi_2 s\psi_3 & s\psi_2 c\psi_3 & s\psi_2 L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

$$\frac{\partial x_{i1}^e}{\partial \varphi_{i3}} = -\eta_{i2} - \varphi_{i3}\eta_{i1}, \qquad \frac{\partial x_{i2}^e}{\partial \varphi_{i3}} = \eta_{i1} - \varphi_{i3}\eta_{i2}, \qquad \frac{\partial x_{i3}^e}{\partial \varphi_{i3}} = \varphi_{i1}\eta_{i1} + \varphi_{i2}\eta_{i2} \tag{35c}$$

Substituting these into Eq. (31) and then into Eq. (25), the expression of the Lagrange equation component resulting from the potential energy of the SDEs can be written as

$$\frac{\partial V}{\partial \boldsymbol{q}} = \boldsymbol{K} \cdot \boldsymbol{q} + \Omega(\boldsymbol{q}) \tag{36}$$

where $\boldsymbol{K}$ is the matrix with constant coefficients and $\Omega(\boldsymbol{q})$ is a vector with elements that are quadratic forms of $\boldsymbol{q}$.

In this context it is worth stressing that elements of the matrix $\boldsymbol{K}$ are identical with those in Ref. 3. However, elements of the vector $\Omega(\boldsymbol{q})$ contain nonlinear components. If the vector $\Omega(\boldsymbol{q})$ is neglected, the linear model is obtained, which does not take into account an influence of centrifugal forces on deflections of a flexible link.

C. Energy Dissipation

Following the procedure from Ref. 3 and considering the linear damping model, an expression for a component of the Lagrange equation can be written in the form

$$\frac{\partial D}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{L} \cdot \boldsymbol{q} \tag{37}$$

D. Equations of Motion

Substituting Eqs. (24), (36), and (37) into the Lagrange equation, the following form of the equations of motion is obtained:

$$\boldsymbol{M}(t) \cdot \ddot{\boldsymbol{q}} + [\boldsymbol{B}(t) + \boldsymbol{L}] \cdot \dot{\boldsymbol{q}} + [\boldsymbol{C}(t) + \boldsymbol{K}] \cdot \boldsymbol{q} + \boldsymbol{F} + \Omega = \boldsymbol{Q} \tag{38}$$

From the above, the following special cases can be considered: 1) linear vibrations about static equilibrum ($\boldsymbol{B} = \boldsymbol{C} = \boldsymbol{0}, \boldsymbol{F} = \boldsymbol{0}, \Omega = \boldsymbol{0}$), 2) nonlinear vibrations about static equilibrium ($\boldsymbol{B} = \boldsymbol{C} = \boldsymbol{0}, \boldsymbol{F} = \boldsymbol{0}$), 3) motion of the system with a moving base when large deflections of a link and centrifugal forces are neglected ($\Omega = \boldsymbol{0}$), and 4) motion of the system with a moving base when large deflections of a flexible link as well as an influence of centrifugal forces are taken into account.

In the most general case (4) the equations form a set of $6n$ nonlinear differential equations of the second order. Various methods can be used for integrating these equations (e.g., the Newmark method with the iterative procedure, the Runge–Kutta method, and the modal method).

Calculation Example

In order to check correctness of formulas and algorithms formulated, they have been applied to vibration analysis of a flexible link of a manipulator. To verify the results obtained, an example presented in Refs. 1 and 2 has been used (these papers give results obtained using the finite element approach).

A scheme of the manipulator considered is shown in Fig. 5. It is a manipulator with three rotary joints. Links 1 and 2 are assumed to be nondeformable whereas link 3 (the last one), which is a beam consisting of two segments with different cross sections, is flexible.

If angles ψ_1, ψ_2, ψ_3 are known functions of time, then the motion of link 3 can be treated as motion of a flexible body with a known base movement. Thus, the formulas derived above can be applied.

For the manipulator considered the matrix $\boldsymbol{A}_0$ and its derivatives $\dot{\boldsymbol{A}}_0$ and $\ddot{\boldsymbol{A}}_0$ can be easily calculated. Taking into account parameters given in the Table 1 and following the procedure given by Craig,[5] the matrix $\boldsymbol{A}_0$ takes the form

The matrices $\dot{\boldsymbol{A}}_0(t)$ and $\ddot{\boldsymbol{A}}_0(t)$, which are necessary for the use of the formulas given in the previous section, can be calculated by deriving the elements of $\boldsymbol{A}_0$.

In the case considered ${}_{i'}^{r}\boldsymbol{T}$ are matrices with constant coefficients and, according to Eq. (10), have the following simple form:

$${}_{i'}^{r}\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x_{i3}^{(c)} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad x_{i3}^{(c)} = \text{const} \tag{40}$$

It is due to the fact that the mass centers of RFEs of the undeformed beam lie on axis 3 of the base frame $\{r\}$.

Having calculated ${}_{i'}^{r}\boldsymbol{T}$ and $\boldsymbol{A}_0, \dot{\boldsymbol{A}}_0, \ddot{\boldsymbol{A}}_0$ expressions from Eq. (18) can be calculated:

$$\dot{\boldsymbol{A}}_{i0} = \dot{\boldsymbol{A}}_0 \cdot {}_{i'}^{r}\boldsymbol{T}, \qquad \ddot{\boldsymbol{A}}_{i0} = \ddot{\boldsymbol{A}}_0 \cdot {}_{i'}^{r}\boldsymbol{T} \tag{41}$$

According to Ref. 3, the beam in question is primarily divided into n equal-size elements of length Δ (Fig. 6). The bending axis of the beam with relation to the central axis 3 is displaced through e in the direction of axis 2.

The following parameters of the beam are given in Ref. 1: $E = 6.8950$ N/m^2, $G = 2.6519$ N/m^2, $\rho = 2766.67$ kg/m^3 for both segments. For the segment $B_1 = 2.667$ m there are

$$J_2^{(1)} = \int_F x_1^2 \, \mathrm{d}F = 1.50 \times 10^{-7} \, \mathrm{m}^4$$

$$J_1^{(1)} = \int_F x_2^2 \, \mathrm{d}F = 1.50 \times 10^{-7} \, \mathrm{m}^4$$

$$F^{(1)} = 3.84 \times 10^{-4} \, \mathrm{m}^2$$

$$a_1^{(1)} = 2.09, \quad a_1^{(2)} = 3.174, \quad \kappa^{(1)} = 2.200 \times 10^{-7} \, \mathrm{m}^4$$

$$e^{(1)} = 0$$

For the segment $B_2 = 5.333$ m there are

$$J_2^{(2)} = 4.8746 \times 10^{-9} \, \mathrm{m}^4, \qquad J_1^{(2)} = 8.2181 \times 10^{-9} \, \mathrm{m}^4$$

$$F^{(2)} = 7.30 \times 10^{-5} \, \mathrm{m}^2$$

$$a_2^{(1)} = 2.09, \qquad a_2^{(2)} = 1.52, \qquad \kappa^{(2)} = 2.433 \times 10^{-11} \, \mathrm{m}^4$$

$$e^{(2)} = e = 0.01875 \, \mathrm{m}$$

Quantities a_1, a_2, and κ are coefficients correcting shear and torsional stiffness, which depend on the shape of a beam cross section.[3]

The quantity Δ from Fig. 6 is defined as

$$\Delta = (B_1 + B_2)/n \tag{42}$$

where n is the number of RFEs (of link 3).

To avoid some problems caused by the change of cross section of the beam in question, in view of the fact that $B_2 = 2B_1$, the assumption $n = 3k$ assures constant stiffness for the elements of the primary division. Thus, the length of elements can be calculated from the formulas[3]

$$l_0 = l_n = \tfrac{1}{2}\Delta, \qquad l_i = \Delta \qquad i = 1, 2, \ldots, n \tag{43}$$

Table 1 Denavit–Hartenberg parameters of manipulator considered

	$i = 1$	$i = 2$	$i = 3$
a_{i-1}	0	L_1	L_2
α_{i-1}	0	$\frac{1}{2}\pi$	$\frac{3}{2}\pi$
d_i	0	0	0
Θ_i	φ_1	φ_2	φ_3

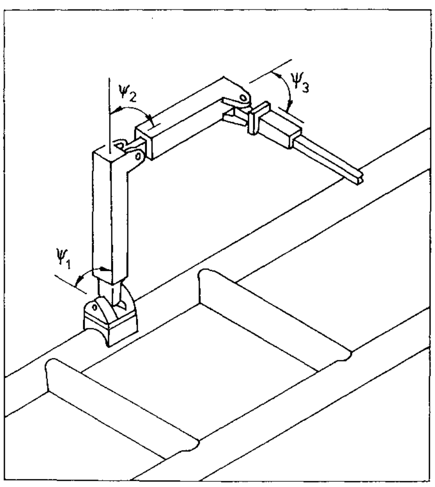

Fig. 5 Scheme of manipulator considered.

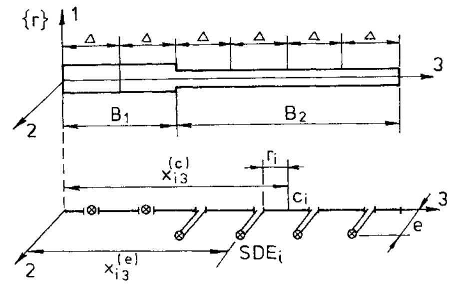

Fig. 6 Division of link 3 into RFEs and SDEs for n = 6.

However, SDEs can be placed in points with the following coordinates:

$$x_{i3}^{(e)} = \tfrac{1}{2}\Delta + (i - 1)\Delta \qquad i = 1, 2, \ldots, n/1 \tag{44}$$

Because the relative motion is performed only by RFEs $1/n$ (the RFE$_0$ is firmly connected with the frame $\{r\}$), it is necessary to define mass parameters of these elements and parameters of SDEs $1 - n$. It is worth pointing out that mass centers of the RFEs $1 - (k - 1)$ and $(k + 1) - n$ lie exactly in the middle of the element length. However, in view of the change of cross section of the beam, the center of the RFE k is displaced.

The following can be easily obtained:

$$x_{i3}^{(c)} = i\Delta \qquad i = 1, 2, \ldots, k - 1, k + 1, \ldots, n - 1 \tag{45a}$$

$$x_{k3}^{(c)} = \tfrac{1}{2}\Delta + (k - 1)\Delta + r_k \qquad r_k = \frac{F^{(1)} + 3F^{(2)}}{4(F^{(1)} + F^{(2)})}\Delta \tag{45b}$$

$$x_{n3}^{(c)} = (n - 1)\Delta + \tfrac{1}{4}\Delta \tag{45c}$$

The mass parameters $(h_{i1}, h_{i2}, h_{i3}, h_{i4})$ calculated from the formulas given in Ref. 3 are defined in Table 2. Stiffness coefficients[3] c_{i1}/c_{i3} and k_{i1}/k_{i3} of SDEs $1/n$, which are necessary to define the stiffness matrix $\boldsymbol{K}$ in Eq. (36), are calculated from formulas given in Table 3.

Results

To carry out the calculations, a software package called BEAM has been worked out based on algorithms described in previous sections. Results obtained for the nonlinear model, which takes into account the vector Ω from Eq. (36), are discussed below. In Ref. 1 two kinds of input functions are considered.

WI Deployment Process

In this case functions $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ change according to the following:

$$\psi_1(t) = \begin{cases} \pi - \dfrac{\pi}{2T}\left(t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right), & 0 < t \le T \\[2ex] \dfrac{1}{2}\pi, & t > T \end{cases}$$

$$\psi_2(t) = \begin{cases} \pi - \dfrac{3\pi}{4T}\left(t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right), & 0 < t \le T \\[2ex] \dfrac{1}{4}\pi, & t > T \end{cases}$$

$$\psi_3(t) = \begin{cases} \pi - \dfrac{\pi}{T}\left(t - \dfrac{T}{2\pi}\sin\dfrac{2\pi t}{T}\right), & 0 < t \le T \\[2ex] 0, & t > T \end{cases}$$

An essential feature of this input is a smooth course of the change of angles. The manipulator from the coiled position turns to the working position.

Table 2 Mass parameters of SDEs

	$i = 1, \dots, k-1$	$i = k$	$i = k+1, \dots, n-1$	$i = n$
h_{i1}	$h_{i4}\dfrac{J_2^{(1)}}{F^{(1)}}$	$m_1\dfrac{J_2^{(1)}}{F^{(1)}} + m_2\dfrac{J_2^{(2)}}{F^{(2)}}$	$h_{i4}\dfrac{J_2^{(2)}}{F^{(2)}}$	$h_{i4}\dfrac{J_2^{(2)}}{F^{(2)}}$
h_{i2}	$h_{i4}\dfrac{J_1^{(1)}}{F^{(1)}}$	$m_1\dfrac{J_1^{(1)}}{F^{(1)}} + m_2\dfrac{J_1^{(2)}}{F^{(2)}}$	$h_{i4}\dfrac{J_1^{(2)}}{F^{(2)}}$	$h_{i4}\dfrac{J_1^{(2)}}{F^{(2)}}$
h_{i3}	$h_{i4}\dfrac{\Delta^2}{12}$	$m_1\dfrac{\Delta^2}{12} + m_1\left(r_k - \dfrac{\Delta}{4}\right)^2 + m_2\dfrac{\Delta^2}{48} + m_2\left(r_k - \dfrac{3}{4}\Delta\right)^2$	$h_{i4}\dfrac{\Delta^2}{12}$	$h_{i4}\dfrac{\Delta^2}{48}$
h_{i4}	$\rho F^{(1)}\Delta$	$m_1 + m_2$, $m_1 = \rho F^{(1)}\dfrac{\Delta}{2}$, $m_2 = \rho F^{(2)}\dfrac{\Delta}{2}$	$\rho F^{(2)}\Delta$	$\rho F^{(2)}\dfrac{\Delta}{2}$

Table 3 Values of stiffness coefficients

	c_{i1}	c_{i2}	c_{i3}	k_{i1}	k_{i2}	k_{i3}
$i = 1, \dots, k$	$\dfrac{GJ_1^{(1)}}{a_1^{(1)}\Delta}$	$\dfrac{GJ_2^{(1)}}{a_2^{(1)}\Delta}$	$\dfrac{EF^{(1)}}{\Delta}$	$\dfrac{EJ_1^{(1)}}{\Delta}$	$\dfrac{EJ_2^{(1)}}{\Delta}$	$\dfrac{G\kappa^{(1)}}{\Delta}$
$i = k+1, \dots, n$	$\dfrac{GJ_1^{(1)}}{a_1^{(1)}\Delta}$	$\dfrac{GJ_2^{(2)}}{a_2^{(2)}\Delta}$	$\dfrac{EF^{(2)}}{\Delta}$	$\dfrac{EJ_1^{(2)}}{\Delta}$	$\dfrac{EJ_2^{(2)}}{\Delta}$	$\dfrac{G\kappa^{(1)}}{\Delta}$

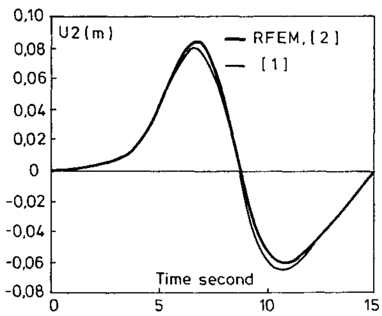

Fig. 7 Displacement U2 of end of link 3 (WI).

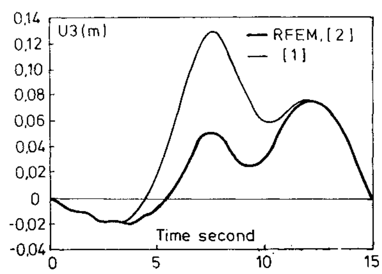

Fig. 8 Displacement U3 of end of link 3 (WI).

WII Spin-Up Maneuver

In this case functions $\psi_1(t)$ and $\psi_3(t)$ are constant; however, an angle $\psi_2(t)$ changes according to the formulas

$$\psi_1(t) = \tfrac{1}{2}\pi, \qquad \psi_3(t) = 0$$

$$\psi_2(t) = \begin{cases} \dfrac{6}{T}\left[\dfrac{t^2}{2} + \left(\dfrac{T}{2\pi}\right)^2\left(\cos\dfrac{2\pi t}{T} - 1\right)\right], & 0 < t \le T \\ 6t - 45, & t > T \end{cases}$$

It is worth stressing that, as opposed to the previous case, the velocity $\dot{\psi}_2(t)$ is not equal to zero as the time $T = 15$ s passes but

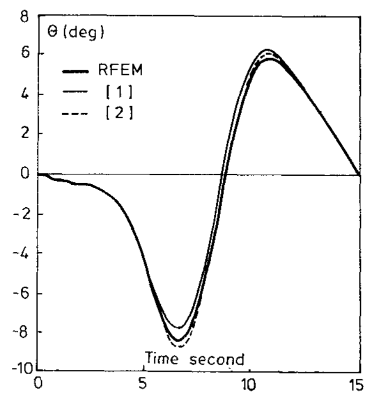

Fig. 9 Twist angle Θ of end of link 3 (WI).

has a constant, rather large magnitude. According to Kane et al.[1] this kind of input can be met in connection with the operation of helicopter rotor blades.

For data of the manipulator given above calculations for both inputs have been carried out. Graphs discussed below present deflections and torsion angles of the end of link 3. Denotations are used as in Ref. 1, and in this connection the following takes place:

$$U_2 = -q_{n,1} - \tfrac{1}{2}q_{n,5}l_n, \qquad U_3 = -q_{n,2} - \tfrac{1}{2}q_{n,4}l_n, \qquad \theta = q_{n,6}$$

where $q_{n,1} - q_{n,6}$ are generalized coordinates of RFE and l_n is the length of RFE_n.

Figures 7–9 show the comparison of the results obtained using the method described with those presented elsewhere[1,2] for the input denoted as WI. A very good correspondence has been achieved with the results from Ref. 2, which were obtained using the finite element approach. In the case of the other kind of input WII (Figs. 10–12) results obtained are also closer to those from Ref. 2 than those from Ref. 1. The comparison of elongation of link 3 along the axis x_{r3} is shown in Fig. 13. Again, the shape of results obtained is approximately that of Ref. 10. The results presented above have been obtained without the modal method, using only the Runge–Kutta method for integrating Eqs. (38). Results shown in Figs. 7–13

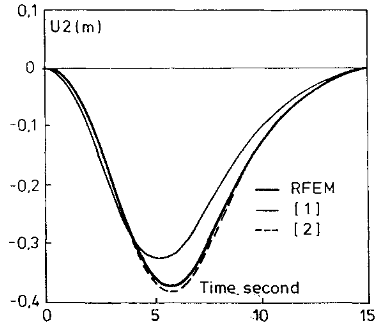

Fig. 10 Displacement U2 of end of link 3 (WII).

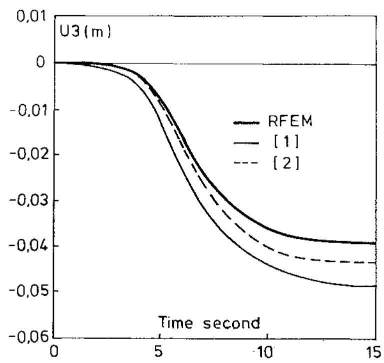

Fig. 11 Displacement U3 of end of link 3 (WII).

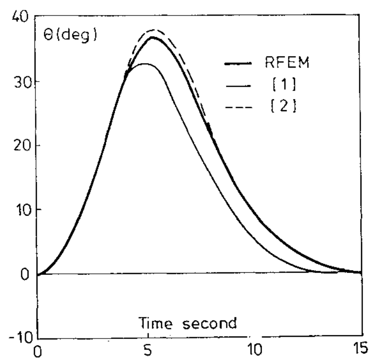

Fig. 12 Twist angle Θ of end of link 3 (WII).

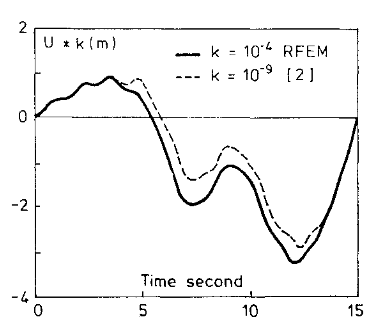

Fig. 13 Longitudinal deflection *U*.

have been obtained for $n = 6$, that is, when the flexible link is divided into seven RFEs and six SDEs.

Conclusions

This paper describes the application of the RFEM to modeling of flexible systems with changing configuration. It is assumed that the base motion is known.

Calculation results for the model presented allow for formulation of the following conclusions:

1) The RFEM is a method with simple physical interpretation enabling obtainment of reliable results not only in the case of vibration analysis about static equilibrium[3] but also for systems with changing configuration.

2) The use of the RFEM allow us to take into account large deflections of the beamlike elements as well as an influence of centrifugal forces on link deflections.

3) By leaving out of consideration the nonlinear components in the equation of motion, the linear model can be easily obtained, and it can be used for modeling of many systems in the case of smooth transportation motion.

The main differences between the method presented and the generally used FEM are as follows:

1) In the formulation of mass parameters of the system, the method presented here takes into account all mass parameters of RFEs. Consideration of their rotary inertia seems to be an advantage of this method in relation to the FEM.

2) In the definition of stiffness parameters of the system, the FEM reflects elastic features of the system considerably better (Fig. 14, longitudinal deflections).

3) In order to derive the equation of motion, the RFEM uses formalism for dynamics of manipulators with rigid links. Thus, in a particular case ($n = 0$) motion of the system with rigid links can be obtained.

It has not been the intention of the authors to prove that the RFEM is better or worse than the FEM. It is simply a different method. Despite its simplicity, the method gives good results in modeling of complex mechanical systems and deserves major popularity. At present further work is being carried out on application of the method to modeling of systems with a changing configuration for more general cases when vibrations of a flexible link influence the base motion.

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